Sheet 2

Problem 1.
We throw n balls uniformly at random into n bins. Show that for large n no bin contains more than $c \ln n / \ln \ln n$ balls for some constant c with probability at least $1 - 1/n$. Hint: First use the Chernoff bound for one bin and then apply the union bound.

Problem 2.
Theorem 4.7 from the book by Motwani and Raghavan for the randomized packet routing algorithm states that with probability at least $1 - 1/N$, every packet reaches its destination in $14n$ or fewer steps. Show that the expected number of steps within which all packets are delivered is at most $15n$.

Problem 3.
Let $X_0 = 0$ and for $j \geq 0$ let X_{j+1} be chosen uniformly over the real interval $[X_j, 1]$. Show that, for $k \geq 0$, the sequence

$$Y_k = 2^k(1 - X_k)$$

is a martingale.

Problem 4.
Consider an urn that initially contains b black balls and w white balls. We perform a sequence of random selections from this urn, where at each step the chosen ball is replaced by c balls of the same color. Let X_i denote the fraction of black balls in the urn after the i-th trial. Show that the sequence $X_0, X_1, \ldots$ is a martingale.

Problem 5.
We are given a Erds–Rnyi random graph $G = (V, E) = G(n, p)$ on nodes $V = \{v_1, \ldots, v_n\}$. The chromatic number $\chi(G)$ is the minimum number of colors needed in order to color all vertices of the graph so that no adjacent vertices have the same color.
Give tail bounds on $\chi(G)$ using a *vertex exposure martingale* as follows. For $1 \leq i \leq n$, let G_i be the induced subgraph on $\{v_1, \ldots, v_i\}$. Let $Y_0 = E[\chi(G)]$ and define Y_i as the (conditional) expectation of $\chi(G)$, conditioned by the knowledge of the edges in G_i, that is, $Y_i = \mathrm{E}\left[\chi(G) | G_i, G_{i-1}, \ldots, G_1\right]$. Show that

$$P[|\chi(G) - \mathrm{E}\left[\chi(G)\right]| \geq \lambda\sqrt{n}] \leq 2e^{-2\lambda^2} \ .$$