Assignment Sheet 1

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Exercise 1.
This problem shows that Markov's inequality is tight. Given a positive integer k, describe a random variable X that assumes only nonnegative values such that

$$\Pr[X \geq k\mathbb{E}[X]] = \frac{1}{k} \ .$$

Can you give an example that shows that Chebyshev's inequality is tight?

Exercise 2.
Consider an experiment where you throw a six-sided die n times. Let X be the random variable indicating the number of times that a 6 occurs. Define $p := \Pr[X \geq n/4]$ as the probability for the event that at least one fourth of all throws is a 6. Compare the best upper bounds on p that you can obtain using Markov's inequality, Chebyshev's inequality, and Chernoff bounds.

Exercise 3.
In the 2017 Austrian National Council Elections, the Green Party received 192,638 votes out of a total of 5,120,879 votes. In order to be allowed representations at the parliament, 4% of the votes would have been required. Assume that a vote for the Green Party is misrecorded with a probability of $p = 0.001$. Use a Chernoff bound to bound the probability, that the Green Party lost the election due to misrecorded votes.

Exercise 4.
In the lecture, we proved the standard Chernoff bound for the sum $X = \sum_{i=1}^{n} X_i$ of n independent 0-1 random variables X_i. That bound requires the exact expectation $\mu := \mathbb{E}[X]$, which is often not available. Prove the following generalized Chernoff bound which holds for an arbitrary upper bound $\mu_H \geq \mu$ on the expectation:

$$\Pr[X \geq (1+\delta) \cdot \mu_H] \leq \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}} \right)^{\mu_H} .$$

Exercise 5.

As before, let $X_1, \ldots, X_n$ be independent 0-1 random variables with $\Pr[X_i] = p_i$. Let $X = \sum_{i=1}^{n} X_i$ with expected value $\mathbb{E}[X] = \mu$. Show that the following slightly looser but much more convenient Chernoff bound holds for any $0 < \delta \leq 1$.

$$\Pr[X \geq (1+\delta)\mu] \leq e^{-\delta^2 \mu/3}$$

Exercise 6.

The HVV Transportation Referendum is a poll that asks Hamburg's population whether they support an increased ticket price to extend the transportation system. Let p denote the actual (currently unknown) fraction of people who support this referendum. Our goal is to get an estimate X of p that is within $\phi \in (0, 1)$ of the true value with a sufficiently high probability. That is, we want

$$\Pr[|X - p| \leq \phi] > 1 - \varepsilon,$$

where $\phi, \varepsilon \in (0, 1)$ are small constants. We compute the estimate X by choosing N people from Hamburg uniformly at random and set X to the fraction of these people that support the referendum. How large should we choose N in order to guarantee the above error probability? Use Chernoff bounds to find an expression for N (in terms of ϕ and ε). How large should we choose N in order to guarantee that X is with 95% probability correct up to $\pm 2\%$?