Assignment Sheet 4

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November 29, 2017

Exercise 11. *Doob martingale*

In the lecture we implicitly considered the *Doob sequence*. The Doob sequence of a function f with respect to a sequence of random variables $X_1, X_2, \ldots, X_n$ is defined by

$$Y_i \coloneqq \mathbb{E}[f \mid X_1, X_2, \ldots, X_i], \quad 0 \leq i \leq n.$$

In particular, we have $Y_0 = \mathbb{E}[f]$ and $Y_n = f(X_1, X_2, \ldots, X_n)$. Prove that the Doob sequence is a martingale with respect to $X_1, X_2, \ldots, X_n$.

Exercise 12. *Pólya urn*

Consider an urn that initially contains b black balls and w white balls. We perform a sequence of random selections from this urn, where at each step the chosen ball is replaced by c balls of the same color. Let X_i denote the fraction of black balls in the urn after the ith trial. Show that the sequence $X_0, X_1, \ldots$ is a martingale.

Exercise 13. *Vertex exposure martingale*

We are given a Erdős–Rényi random graph $G = (V, E) = G(n, p)$ on nodes $V = \{ v_1, \ldots, v_n \}$. The chromatic number $\chi(G)$ is the minimum number of colors needed in order to color all vertices of the graph so that no adjacent vertices have the same color.

Give tail bounds on $\chi(G)$ using a *vertex exposure martingale* as follows. For $1 \leq i \leq n$, let G_i be the induced subgraph on $\{ v_1, \ldots, v_i \}$. Let $Y_0 = \mathbb{E}[\chi(G)]$ and define Y_i as the (conditional) expectation of $\chi(G)$, conditioned by the knowledge of the edges in G_i, that is, $Y_i = \mathbb{E}[\chi(G) \mid G_i, G_{i-1}, \ldots, G_1]$. Show that

$$\Pr\left[|\chi(G) - \mathbb{E}[\chi(G)]| \geq \lambda\sqrt{n}\right] \leq 2e^{-2\lambda^2} \ .$$