Assignment Sheet 5

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Exercise 14.
Consider the infinite lattice $\mathbb{Z} \times \mathbb{Z}$. A pebble starting from the origin walks at random, each time moving to one of the four neighbors with equal probability. Let the random variable X_i denote the Manhattan distance (i.e., number of hops) between the pebble and the origin after the i-th step.

(a) Show that this defines a martingale sequence satisfying the bounded difference condition.
(b) What is the probability that after n steps, the pebble is at distance $> 100\sqrt{n}$ from the origin?

Exercise 15.
Consider random variables $X_1, X_2, \dots$ and assume that the partial sums $S_n := \sum_{i=1}^{n} X_i$ form a martingale (with respect to $X_1, X_2, \dots$). Prove that $\mathbb{E}[X_i \cdot X_j] = 0$ if $i \neq j$.

Exercise 16.
A parking-lot attendant has mixed up n keys for n cars. The n car owners arrive together. The attendant gives each owner a key according to a permutation chosen uniformly at random from all permutations. If an owner receives the key to his car, he takes it and leaves; otherwise, he returns the key to the attendant. The attendant now repeats the process with the remaining keys and car owners. This continues until all owners receive the keys to their cars. Let R be the number of rounds until all car owners receive the keys to their cars. We want to compute $\mathbb{E}[R]$. Let X_i be the number of owners who receive their car keys in the i-th round. Prove that

$$Y_i = \sum_{j=1}^{i} (X_i - \mathbb{E}[X_i \mid X_1, \dots, X_{i-1}])$$

is a martingale. Use the martingale stopping theorem to compute $\mathbb{E}[R]$.

Exercise 17.
Alice and Bob play each other in a checkers tournament, where the first player to win four games wins the match. The players are evenly matched, so the probability that each player wins each game is $1/2$, independent of all other games. The number of minutes for each game is uniformly distributed over the integers in the range $[30, 60]$, again independent of other games. What is the expected time they spend playing the match?

Exercise 18.
Consider the following extremely inefficient algorithm for sorting n numbers in increasing order. Start by choosing one of the n numbers uniformly at random, and placing it first. Then choose one of the remaining $n-1$ numbers uniformly at random, and place it second. If the second number is smaller than the first, start over again from the beginning. Otherwise, next choose one of the remaining $n-2$ numbers uniformly at random, place it third, and so on. The algorithm starts over from the beginning whenever it finds that the kth item placed is smaller than the $(k-1)$th item. Determine the expected number of times the algorithm tries to place a number, assuming that the input consists of n distinct numbers.