Randomised Approximation

Sheet 1

Due date: No deadline

Exercise 1.
Four out of ten people have rose flowers. If we choose three persons at random, what will be the probability of selecting at least one with rose flower?

Solution 1.
The probablity of this event is the complement of having no one with rose flower in three selections(without replacement) which is $1 - \frac{6}{10} \cdot \frac{5}{9} \cdot \frac{4}{8}$

Exercise 2.
There are x red and 6 blue balls in a bin. We choose two balls at random. If the probability of having same colors in the outcome is 1/2, then what would be x?

Solution 2.
probability of choosing two red balls $= \frac{x}{x+6} \cdot \frac{x-1}{x+5}$.

probability of choosing two blue balls $= \frac{6}{x+6} \cdot \frac{5}{x+5}$.

Therefore, $\frac{x}{x+6} \cdot \frac{x-1}{x+5} + \frac{6}{x+6} \cdot \frac{5}{x+5} = \frac{1}{2}$. Simplifying this results in $x^2 - 13x + 30 = 0$ which gives us $x = 10$ or $x = 3$.

Exercise 3.
We flip a fair coin 10 times. Find the probability of the following events.

(a) The number of heads and the number of tails are equal.

(b) There are more heads than tails.

(c) The ith flip and the $(11 - i)$th flips are the same for $i = 1, \cdots, 5$.

(d) We flip at least four consecutive heads.

Solution 3.

(a) In this event we have 5 times heads and 5 times tails. First we choose 5 times out of ten for the heads which is $\binom{10}{5}$, this 5 heads can happen with probability $(1/2)^5$ and then, we choose 5 times for tails out of the remaining 5 times which is $\binom{5}{5}$. This case can happen with probability $(1/2)^5$. After all, this event can happen with probability $E = \binom{10}{5} \cdot (1/2)^5 \cdot \binom{5}{5} \cdot (1/2)^5$.

(b) Three cases in general may happen. Either the number of tails is equal to the number of heads, or the number of heads is more than the number of tails or the number of tails is more than the number of heads. Note that the coin is fair. Therefore, the probability of two last cases is equal. Hence, this event can happen with probability $(1 - E)/2$. (E is the probability of having the same number of heads and tails which we computed in the first event.)

(c) This event is having palindrome. For this event, we first fix one permutation for the first 5 times. Then, based on that we copy this outcome for five remaining times according to the palindrome property (For example, after fixing the first five times, we copy the first time's outcome for the 10th time and so on). Obviously, we can fix the first 5 times with probability $(1/2)^5$ which is the answer.

(d) Clearly, $Pr[\text{flip} \geq 4 \text{ consecutive heads}] = 1 - Pr[\text{flip} < 4 \text{ consecutive heads}]$. Notice that there are four sequences that do not lead to four consecutive heads: $P[T] = 1/2$, $P[HT] = 1/2^2$, $P[HHT] = 1/2^3$ and $P[HHHT] = 1/2^4$. Therefore, we can set up a recursion for k flips where P_k is the probability of not observing four consecutive heads in k flips. Note that, $P_0 = P_1 = P_2 = P_3 = 1$, in order to allow sequence ending in heads. Hence, we have

$$P_k = (1/2)P_{k-1} + (1/4)P_{k-2} + (1/8)P_{k-3} + (1/16)P_{k-4}.$$

Therefore, $P_{10} = 0.245$.

Exercise 4.
We shuffle a standard deck of cards, obtaining a permutation that is uniform over all 52! possible permutations. Find the probability of the following events.

(a) The first five cards include at least one ace.

(b) The first two cards are a pair of the same rank.

(c) The first five cards are all diamonds.

(d) The first five cards form a full house (three of one rank and two of another rank).

Solution 4.

Exercise 5.
We have a six-sided dice which comes to side i with probability proportional to i. What is the probability of having even number in one roll?

Solution 5.

Exercise 6.
Suppose that a fair coin is flipped n times. For $k > 0$, find an upper bound on the probability that there is a sequence of $\log_2 n + k$ consecutive heads.

Solution 6.

Exercise 7.
A group of n men enter a restaurant and check their hats. The hat-checker is absent minded, and upon leaving, she redistributes the hats back to the men at random. What is the probability P_n that no man gets his correct hat, and how does P_n behave as n approaches infinity?

Solution 7.

If you have any question regarding the problems, please do not hesitate to contact us.