Randomised Algorithms

Sheet 10

Due date: 09.02.2021

Exercise 1.
Let G be a 3-colorable graph.

(a) Show that there exists a coloring of the graph with two colors such that no triangle is monochromatic. (A triangle of a graph G is a subgraph of G with three vertices, which are all adjacent to each other.)

(b) Consider the following algorithm for coloring the vertices of G with two colors so that no triangle is monochromatic. The algorithm begins with an arbitrary 2-coloring of G. While there are any monochromatic triangles in G, the algorithm chooses one such triangle and changes the color of a randomly chosen vertex of that triangle. Derive an upper bound on the expected number of such recoloring steps before the algorithm finds a 2-coloring with the desired property.

Exercise 2.
Let X be a Poisson random variable with mean μ representing the number of errors on a page of some book. Each error is independently a grammatical error with probability p and a spelling error with probability $1 - p$. If Y and Z are random variables representing the number of grammatical and spelling errors (respectively) on a page of that book, prove that Y and Z are Poisson random variables with means μp and $\mu(1 - P)$, respectively. Also, prove that Y and Z are independent.

Exercise 3.
Suppose that balls are thrown randomly into n bin. Show, for some constant c_1, that is there are $c_1\sqrt{n}$ balls then the probability that no two land in the same bins is at most $1/e$. Similarly, show for some constant c_2 (and sufficiently large n) that, if there are $c_2\sqrt{n}$ balls, then the probability that no two land in the same bin is at least 1/2. make these constants as close to optimal as possible.

Hint: You may want to use the facts that $e^{-x} \geq 1 - x$ and $e^{-x-x^2} \leq 1 - x$ for $x \leq \frac{1}{2}$.

Exercise 4.
Consider the probability that every bin receives exactly one ball when n balls are thrown randomly into n bins.

(a) Give an upper bound on this probability using the Poisson approximation.

(b) Determine the exact probability of this event.

(c) Show that these two probabilities differ by a multiplicative factor that equals the probability that a Poisson random variable with parameter n takes on the value n. Explain why this is implied by Theorem 5.6 from the book by Mitzenmacher and Upfal.

Exercise 5.

If you have any question regarding the problems, please do not hesitate to contact us.