Randomised Algorithms

Sheet 11

Due date: 16.02.2021

Exercise 1.
Consider an undirected (not bipartite)graph $G(V, E)$ with $|V| = n$. Show that a random walk on G converges to a stationary distribution π where $\pi_i = \frac{d_i}{2|E|}$ for $n \geq i \geq 1$.

Exercise 2.
Consider the same given graph as last problem. Let $h_{u,v}$ be the expected time a random walk needs to reach node v starting from node u in graph G. Show that if $(u, v) \in E$ then $h_{u,v} + h_{v,u}$ is at most $2|E|$.

Exercise 3.

(a) The cover time of a graph $G = (V, E)$ is the maximum over all vertices $v \in V$ of the expected time to visit all of the nodes in the graph by a random walk starting from v. Show that the cover time of G is bounded above by $2|E|(|V| - 1)$.

(b) There is a relation between the hitting time and cover time a graph. How can you relate this two? ('You may use the argument in lemma 7.16 of Mitzenmacher book!)

Exercise 4.
The wrapped butterfly network has $N = n2^n$ nodes. The nodes are arranged in n columns and 2^n rows. A node's address is a pair (x, r), where $1 \leq x \leq 2^n$ is the row number and $0 \leq r \leq n - 1$ is the column number of the node. Node (x, r) is connected to node (y, s) iff $s = r + 1 \mod n$ and either

- $x = y$ (the 'direct' edge); or
- x and y differ in precisely the sth bit in their binary representation (the 'flip' edge).

Consider the following random permutation routing algorithm on the butterfly.

Algorithm 1: Three Phase Routing Algorithm

For a packet sent from node (x, r) to node (y, s) do the following.
Phase I- Choose a random $w \in [1, \cdots, 2^n]$. Route the packet from node (x, r) to node (w, r) using the bit-fixing route.
Phase II- Route the packet to node (w, s) using direct edge.
Phase II- Route the packet from node (w, s) to node (y, s) using bit-fixing route.

As you can see we can not use our analysis for hypercube here because we cannot simply bound the number of active packets that possibly traverse edges of a path. To do the analysis we need to define some priority. Therefore, we have the following rules:

(a) The priority of a packet traversing an edge is $(i - 1)n + t$, where i is the current phase of the packet and t in the number of edge traversals the packet has already executed in this phase.

(b) If at any step more than one packet is available to traverse an edge, the packet with the smallest priority number is sent first.

It can be shown the above algorithm guarantees with high probability all packets reach their destinations in parallel time $O(n) = O(\log N)$. Using the similar argument as proof for randomized routing algorithm for hypercube it can be proved that the Phase-II needs at most $5n$ steps. The two remaining phases are symmetric. Hence, we just try to calculate the time for the first phase. Before that, we need to define the following tool:

Definition 1. *A **delay sequence** for an execution of Phase I is a sequence of n edges $e_1, \cdots, e_n$ such that either $e_i = e_{i+1}$ or e_{i+1} is an outgoing edge from the end vertex of e_i. The sequence $e_1, \cdots, e_n$ has the further property that e_i is (one of) the last edges to transmit packets with priority up to i among e_{i+1} and the two incoming edges of e_{i+1}.*

For a given execution of Phase I and delay sequence $e_1, \cdots, e_n$, let t_i be the number of packets with priority i sent through edge e_i. Let T_i be the time that edge e_i finishes sending all packets with priority number up to i, so that T_n is the earliest time at which all packets passing through e_n during Phase I have passed through it. Then **prove that:**

(a) $T_n \leq \sum_{i=1}^{n} t_i$.

(b) If the execution of Phase I takes T steps, then there is a delay sequence for this execution for which $\sum_{i=1}^{n} t_i \geq T$.

(Following this arguments one can also show that the probability of a delay sequence with $T \geq 40n$ is only $O(N^{-1})$, and then showing that the total time for this three phases random algorithm to deliver all packets is $O(n)$ with probability 1-$O(N^{-1})$.)

If you have any question regarding the problems, please do not hesitate to contact us.