Randomised Approximation

Sheet 3

Due date: 24.11.2020

Exercise 1.
Prove that, for any real number c and any discrete random variable X, $Var[cX] = c^2 Var[X]$.

Exercise 2.
Let the random variable X be representable as a sum of random variables $X = \sum_{i=1}^{n} X_i$. Show that, if $E[X_i X_j] = E[X_i]E[X_j]$ for every pair of i and j with $1 \leq i < j \leq n$, then $Var[X] = \sum_{i=1}^{n} Var[X_i]$.

Exercise 3.
Let $n \in N$ and consider the complete graph K_n (a clique of n nodes). Let $k \in N$ be such that $\binom{n}{k} \cdot 2^{-\binom{k}{2}+1} < 1$. Show that there exists an edge-coloring of K_n with two colors such that it has no monochromatic K_k as a subgraph.

Exercise 4.
Consider a restaurant that serves three different dinners for \$12, \$15, and \$20, respectively. We randomly select a couple visiting the restaurant and define the random variable X as the cost of the man's dinner and the random variable Y as the cost of the woman's dinner. The following table describes the corresponding probability distribution:

$\Pr[X = x \wedge Y = y]$	$y = 12$	$y = 15$	$y = 20$
$x = 12$	0.05	0.05	0.10
$x = 15$	0.05	0.10	0.35
$x = 20$	0.00	0.20	0.10

(a) Compute the covariance $Cov[X, Y]$ of X and Y.

(b) What happens if you change the unit from dollar to cent? Is this a desirable effect?

(c) Explain (informally) why the covariance is negative in this example. How can you change the example such that the sign changes?

Exercise 5.
Suppose we have a sequence of items passing by one at a time. We want to maintain a sample of one item with the property that it is uniformly distributed over all the items that we have seen at each step. Moreover, we want to accomplish this without knowing the total number of items in advance or storing all of the items that we see. Consider the following algorithm, which stores just one item in memory at all times. When the first item appears, it is stored in the memory. When the kth item appears, it replaces the item in memory with probability $1/k$. Explain why this algorithm solves the problem.

Exercise 6.

Consider a simplified version of roulette in which you wager x dollars on either red or black. The wheel is spun, and you receive your original wager plus another x dollars if the ball lands on your color; if the ball doesn't land on your color, you lose your wager. Each color occurs independently with probability $1/2$. The following gambling strategy is popular: On the first spin, bet 1 dollar. If you lose, bet 2 dollars on the next spin. In general, if you have lost on the first $k-1$ spins, bet 2^{k-1} dollars on the k-th spin. Argue that you will eventually win a dollar. Now let X be the random variable that measures your maximum loss before winning (i.e., the amount of money you have lost before the play on which you win). Show that $E[X]$ is unbounded. What does it imply about the practicality of this strategy?

If you have any question regarding the problems, please do not hesitate to contact us.