Randomised Algorithms

Sheet 4

Due date: 01.12.2020

Exercise 1.
Consider an experiment where you throw a six-sided die n times. Let X be the random variable indicating the number of times a 6 occurs. Define $p := Pr[X \geq n/4]$ as the probability for the event that at least one fourth of all throw is a 6. Compare the best upper bounds on p that you can obtain using Markov's inequality, Chebyshev's inequality and Chernoff bounds.

Exercise 2.
Let X_i for $1 \leq i \leq n$ be identical independent 0-1 random variable with $E[X_i] = p$. Prove the following bound for lower tail. Consider $S = \sum_1^n X_i$, $\mu = E[S]$ and $0 < \delta < 1$.

$$\Pr(S \leq (1-\delta)\mu) \leq (\tfrac{e^{-\delta}}{(1-\delta)^{1-\delta}})^{\mu}$$

Exercise 3.
Alice and Bob play checkers often. Alice is a better player, hence the probability that she wins any given game is 0.6, independent of all other games. They decide to play a tournament of n games. Bound the probability that Alice loses the tournament using a Chernoff bound.

Exercise 4.
Consider a biased coin with probability $p = 1/3$ of landing heads and probability $2/3$ of landing tails. Suppose the coin is flipped some number n of times, and let X_i be a random variable denoting the i-th flip, where $X_i = 1$ means heads, and $X_i = 0$ means tails. Use the Chernoff bound to determine a value for n so that the probability that more than half of the coin flips come out heads is less that 0.001.

Exercise 5.
We plan to conduct an opinion poll to find out the percentage of people in a community who want its president impeached. Assume that every person answers either yes or no. If the actual fraction of people who want the president impeached is p, we want to find an estimate X of p such that

$$P[|X - p| \leq \epsilon p] > 1 - \delta$$

for a given ϵ and δ, with $0 < \epsilon, \delta < 1$.

We query N people chosen independently and uniformly at random from the community and output the fraction of them who want the president impeached. How large should N be for our result to be a suitable estimator of p? Use Chernoff bounds, and express N in terms of p, ϵ, and δ. Calculate the value of N from your bound if $\epsilon = 0.1$ and $\delta = 0.05$ and if you know that p is between 0.2 and 0.8.

Exercise 6.

A casino is testing a new class of simple slot machines. Each game, the player puts in \$1, and the slot machine is supposed to return either \$3 to the player with probability 4/25, \$100 with probability 1/200, or nothing with all remaining probability. Each game is supposed to be independent of other games. The casino has been surprised to find in testing that the machines have lost \$10,000 over the first million games. Derive a Chernoff bound for the probability of this event.

If you have any question regarding the problems, please do not hesitate to contact us.