Randomised Algorithms

Sheet 5

Due date: 08.12.2020

Exercise 1.
We plan to conduct an opinion poll to find out the percentage of people in a community who want its president impeached. Assume that every person answers either yes or no. If the actual fraction of people who want the president impeached is p, we want to find an estimate X of p such that

$$P[|X - p| \leq \epsilon p] > 1 - \delta$$

for a given ϵ and δ, with $0 < \epsilon, \delta < 1$.

We query N people chosen independently and uniformly at random from the community and output the fraction of them who want the president impeached. How large should N be for our result to be a suitable estimator of p? Use Chernoff bounds, and express N in terms of p, ϵ, and δ. Calculate the value of N from your bound if $\epsilon = 0.1$ and $\delta = 0.05$ and if you know that p is between 0.2 and 0.8.

Exercise 2.
A casino is testing a new class of simple slot machines. Each game, the player puts in \$1, and the slot machine is supposed to return either \$3 to the player with probability 4/25, \$100 with probability 1/200, or nothing with all remaining probability. Each game is supposed to be independent of other games. The casino has been surprised to find in testing that the machines have lost \$10,000 over the first million games. Derive a Chernoff bound for the probability of this event.

Exercise 3.
Consider a collection $X_1, X_2, \cdots, X_n$ of n independent variables chosen uniformly from the set $\{0, 1, 2\}$. Let $X = \sum_{i=1}^{n} X_i$ and $0 < \delta < 1$. Derive a Chernoff bound for $\Pr[X \geq (1 + \delta) \cdot E[X]]$ and $\Pr[X \leq (1 - \delta) \cdot E[X]]$.

Exercise 4.
Consider the bit-fixing algorithm for routing a permutation on the n-cube ($N = 2^n$). Suppose n is even. Write each source node s as the concatenation of two binary string a_s and b_s each of length $n/2$. Let the destination of s's packet be the concatenation of b_s and a_s. Show that this permutation causes the bit-fixing routing algorithm to take $\Omega(\sqrt{N})$

Exercise 5.
Prove the exercise 4.4 (from the randomized-algorithms, Motwani and Raghavan book) which states as follow. Does the statement in Exercise 4.3 imply that for any two packets v_i and v_j, there is at most one queue q such that v_i and v_j are in the queue q at the same step?

Exercise 6.

We throw n balls uniformly at random into n bins. Show that for large n no bin contains more than $c \ln n / \ln \ln n$ balls for some constant c with probability at least $1 - 1/n$. Hint: First use the Chernoff bound for one bin and then apply the union bound.

If you have any question regarding the problems, please do not hesitate to contact us.