Randomised Algorithms

Sheet 6

Due date: 15.12.2020

Exercise 1.
We have seen in the lecture that packet routing in hypercube using randomized algorithm needs fewer than $14n$ steps with probability at least $1-1/N$ to deliver all packets. Show that the expected number of steps within which all packets are delivered is less than $15n$.

Exercise 2.
Consider the following randomized version of $Insertion - sort$, where we first compute a random permutation and then apply the normal $Insertion - sort$ algorithm:

Algorithm 1: Rand-Insertion-sort(A)

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Result: Sorts an array A[1 ··· n] in non-decreasing order
Random Permutation(A);
for j ← 2 to n do
    key ← A[j] ; i ← j-1 ;
    while (j > 0) and (A[i] > key) do
        A[i+1]← A[i]; i ← i-1;
    end
    A[i+1]← key;
end
```

Analyze the expected number of comparisons made by the algorithm. (An asymptotic analysis is not sufficient: You should analyze the exact number of expected comparisons.)

Exercise 3.
Given an $n \times n$ matrix A all of whose entries are 0 and 1. Find a column vector $b \in \{-1, +1\}^n$ that minimize $||Ab||_\infty$. (Hint: For $1 \leq i \leq n$ set $b_i = 1$ with probability 1/2 and the same for $b_i = -1$.)

Exercise 4.
Consider the following weighted version of $MAX - SAT$ problem. Each clause has a positive weight and the goal is to maximize the weights of the satisfied clauses. Show that there is a truth assignment satisfying clauses the sum of whose of weights is at least half of the total weights.

Exercise 5.
Suppose that we have n jobs to distribute among m processors. For simplicity, we assume that m divides n. A job takes 1 step with probability 1 and $k > 1$ with probability 1-p. use Chernoff bounds to determine upper (and lower bounds) that holds with high probability(more than $1 - 1/n^c$ for $c \geq 1$) on when all jobs will be completed if we randomly assign exactly n/m jobs to each processor.

Exercise 6.

You are hosting a web service. Whenever someone visits your website an algorithm called $LV - ALG$ is executed. It has expected running time of 2 seconds.

(a) Give a bound on the probability that the actual running time of $LV - ALG$ exceeds 1 hour.

(b) What is the expected number of visitors before one of them has to wait 1 hour for $LV - ALG$ to finish?

(c) Now consider the following algorithm, which we call $LV - ALG - With - Restart$: Start running $LV - ALG$. If the algorithm terminates within 4 seconds, then we are done and so we stop. But if not, we abort the execution and start all over again. (Thus $LV - ALG$ is repeated until we get a run terminating within 4 seconds.) Give a bound on the probability that the running time of $LV - ALG - With - Restart$ exceeds 2 minutes. (Assume that testing whether algorithm runs for 4 seconds, aborting the execution and restarting does not take any time.)

(d) What is the expected number of visitors before one of them has to wait for more than 2 minutes if we use the $LV - ALG - With - Restarts$?

If you have any question regarding the problems, please do not hesitate to contact us.