Randomised Algorithms

Sheet 7

Due date: 12.01.2021

Exercise 1.
Show sum of a finite number of independent Poisson random variables is a Poisson random variable.

Exercise 2.
Consider Bit-Fixing routing algortihm on Hypercube. View each route as a directed path in the hypercube from the source to the destination. Show that once two routes seperate, they do not rejoin.

Exercise 3.
There are 30 people in a room, what is the probability of the event in which at least two persons have the same birthday? How many people should be there such that with high probability (more than $1 - 1/n^c$ for some constant c) at least two persons have the same birthday?

Exercise 4.
Suppose that you built a Bloom filter for a dictionary of words with $m = 2^b$ bits. A co-worker building an application wants to use your Bloom filter but has only 2^{b-1} bits available. Explain how your colleague can use your Bloom filter to avoid rebuilding a new Bloom filter using the original dictionary of words.

Exercise 5.
Bloom filters can be used to estimate set differences. Suppose you have a set X and I have a set Y, both with n elements. For example, the sets might represent our 100 favorite songs. We both create Bloom filters of our sets, using the same number of bits m and the same k hash functions. Determine the expected number of bits where our Bloom filters differ as a function of m, n, k, $|X \cap Y|$! Explain how this could be used as a tool to find people with the same taste in music more easily than comparing lists of songs directly.

Exercise 6.
Show that $E[X] = E[E[X|Y]]$ where X, Y are random variables. Also, show $Var(X) = E[Var(X|Y)] + Var(E[X|Y])$.

Exercise 7.
Let $X_0 = 0$ and for $j \geq 0$ let X_{j+1} be chosen uniformly over the real interval $[X_j, 1]$. Show that, for $k \geq 0$, the sequence

$$Y_k = 2^k(1 - X_k)$$

is a martingale.

Exercise 8.
Consider an urn that initially contains b black balls and w white balls. We perform a sequence of random selections from this urn, where at each step the chosen ball is replaced by c balls of the same color. Let X_i denote the fraction of black balls in the urn after the i-th trial. Show that the sequence $X_0, X_1, \ldots$ is a martingale.

Exercise 9.
A parking-lot attendant has mixed up n keys for n cars. The n car owners arrive together. The attendant gives each owner a key according to a permutation chosen uniformly at random from all permutations. If an owner receives the key to his car, he takes it and leaves; otherwise, he returns the key to the attendant. The attendant now repeats the process with the remaining keys and car owners. This continues until all owners receive the keys to their cars. Let R be the number of rounds until all car owners receive the keys to their cars. We want to compute $E[R]$. Let X_i be the number of owners who receive their car keys in the i-th round. Prove that

$$Y_i = \sum_{j=1}^{i} (X_j - E[X_j | X_1, \ldots, X_{j-1}])$$

is a martingale. Use the martingale stopping theorem to compute $E[R]$.

Exercise 10.
Consider the following extremely inefficient algorithm for sorting n numbers in increasing order. Start by choosing one of the n numbers uniformly at random, and placing it first. Then choose one of the remaining $n - 1$ numbers uniformly at random, and place it second. If the second number is smaller than the first, start over again from the beginning. Otherwise, next choose one of the remaining $n-2$ numbers uniformly at random, place it third, and so on. The algorithm starts over from the beginning whenever it finds that the k-th item placed is smaller than the $(k - 1)$-th item. Determine the expected number of times the algorithm tries to place a number, assuming that the input consists of n distinct numbers.

If you have any question regarding the problems, please do not hesitate to contact us.