Randomised Algorithms

Sheet 8

Due date: 26.01.2021

Exercise 1.
Consider a process $X_0, X_1, X_2, ...$ with two states. The process is governed by two matrices, P and Q. If the time k is even, the transition probabilities are P. Otherwise, they are Q. Explain why this process does not satisfy the definition of a time-homogeneous Markov chain. Give an equivalent process (with a larger state space) that satisfies the definition.

Exercise 2.
An $n \times n$ matrix P is said to be stochastic if all its entries are non-negative and for each row i, $\sum_i P_{ij} = 1$. It is said to be doubly stochastic if, in addition, $\sum_j P_{ij} = 1$.

Suppose that the transition probability matrix P for a Markov chain is doubly stochastic. Show that the stationary distribution for this Markov chain is necessarily the uniform distribution.

Exercise 3.
Let X be the sum of n independent rolls of a fair die. Show that, for any $k \geq 2$,

$$\lim_{n\to\infty} P[X_n \text{ is divisible by } k] = 1/k \ .$$

Exercise 4.
Consider the gambler‘s ruin problem, where a player plays until he wins l_2 or loses l_1 dollars. Prove the expected number of games played is $l_1 l_2$.

Exercise 5.
Each day, your opinion on a particular political issue is either positive, neutral, or negative. If it is positive today it is neutral or negative tomorrow with equal probability. If it is neutral or negative, it stays the same with probability 0.5, and otherwise it is equally likely to be either of the other two possibilities. Is this a reversible Markov chain?

Exercise 6.
A total of m white and m black balls are distributed into two urns, with m balls per urn. At each step, a ball is randomly selected from each urn and the two balls are interchanged. The state of this Markov chain can be thus described by the number of black balls in urn 1. Prove this chain is reversible.

Exercise 7.
(Question from Lecture)
Consider $\mathcal{Z}_m$ as a random walk on $0, 1, \cdots, m$ such that $Pr(Z_{t+1} = i | Z_t = i-1) = Pr(Z_{t+1} = i | Z_t = i+1) = 1/2$ for $1 \leq i \leq m-1$ with two absorbing barriers at both endpoints. Show the bound of $m^2/4$ on the expected time until the random walk on Z_m hits a barrier (see Slide 7 pg. 40).

(Hint: you may consider two distinct random walks each with one absorbing barrier, then compute the probability of hitting these barriers. Then you may use geometric variable! to compute the expected value you need.)

Exercise 8.

Let G be a 3-colorable graph.

(a) Show that there exists a coloring of the graph with two colors such that no triangle is monochromatic. (A triangle of a graph G is a subgraph of G with three vertices, which are all adjacent to each other.)

(b) Consider the following algorithm for coloring the vertices of G with two colors so that no triangle is monochromatic. The algorithm begins with an arbitrary 2-coloring of G. While there are any monochromatic triangles in G, the algorithm chooses one such triangle and changes the color of a randomly chosen vertex of that triangle. Derive an upper bound on the expected number of such recoloring steps before the algorithm finds a 2-coloring with the desired property.

Exercise 9.

Consider a Markov Chain on states $0, 1, \ldots$ with transition probabilities $P_{ij} = \frac{1}{i+2}$ for $j = 0, 1, \ldots, i, i+1$. Find the stationary distribution for this Markov Chain.

If you have any question regarding the problems, please do not hesitate to contact us.