# Randomised Algorithms

## Sheet 9

### Due date: 02.02.2021

**Exercise 1.**
A total of m white and m black balls are distributed into two urns, with m balls per urn. At each step, a ball is randomly selected from each urn and the two balls are interchanged. The state of this Markov chain can be thus described by the number of black balls in urn 1. Prove this chain is reversible.

**Exercise 2.**
Let $G$ be a 3-colorable graph.

(a) Show that there exists a coloring of the graph with two colors such that no triangle is monochromatic. (A triangle of a graph $G$ is a subgraph of G with three vertices, which are all adjacent to each other.)

(b) Consider the following algorithm for coloring the vertices of $G$ with two colors so that no triangle is monochromatic. The algorithm begins with an arbitrary 2-coloring of $G$. While there are any monochromatic triangles in $G$, the algorithm chooses one such triangle and changes the color of a randomly chosen vertex of that triangle. Derive an upper bound on the expected number of such recoloring steps before the algorithm finds a 2-coloring with the desired property.

**Exercise 3.**
Consider a Markov Chain on states $0, 1, \dots$ with transition probabilities $P_{ij} = \frac{1}{i+2}$ for $j = 0, 1, \dots, i, i+1$. Find the stationary distribution for this Markov Chain.

**Exercise 4.**
Consider we are given a graph $G(V, E)$ with $|E| = n$, maximum degree $\Delta$ and $q$ colors $Q = \{1, \cdots, q\}$. The Goal is to sample proper coloring of $G$ at random which we have shown that in the lecture. You can consider the MC as follow.

- Current state is any proper coloring of $G$.
- Pick a vertex $v \in V$ and a color $c \in Q$ both uniformly at random
- Recolor $v$ with $c$ if this yields a proper coloring, else do nothing

There is one question as follow. Show that Whenever $q \geq \Delta + 2$, then the corresponding Markov chain is irreducible.

**Exercise 5.**
Prove that a Random walk on an undirected graph G is aperiodic if and only if G is not bipartite.

**Exercise 6.**
A queue is a line where costumers wait for service. We examine a model for a bounded queue where time is divided into steps of equal length. At each time step, exactly one of the following happens.

- If the queue has less than $n$ customers, then with probability $\lambda$ a new customer joins the queue.
- If the queue is not empty, then with probability $\mu$ the load if the line is served and leaves the queue.
- With the remaining probability, the queue is unchanged.

Show that $\pi_i = \left(\frac{\lambda}{\mu}\right)^i \cdot \left(1 - \frac{\lambda}{\mu}\right)$.

**Exercise 7.**
Consider a Markov chain on state $\{0, 1, \cdots, n\}$, where for $i < n$ we have $P_{i,i+1} = P_{i,0} = 1/2$. Also, $P_{n,n} = P_{n,0} = 1/2$. This process can be viewed as a random walk on a directed graph with vertices $\{0, 1, \cdots, n\}$, where each vertex has two directed edges: one that returns to 0 and the one that moves to the vertex with next higher number (with a self-loop at vertex n).Find the stationary distribution of this chain. (This example shows that random walks on directed graphs are very different than random walks on undirected graphs.)

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If you have any question regarding the problems, please do not hesitate to contact us.